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# The Gravitational Sector in the Connes-Lott Formulation of the Standard Model



**A.H. Chamseddine, J. Fröhlich, O. Grandjean**

Theoretical Physics, ETH-Hönggerberg
CH–8093 Zürich, Switzerland

**Abstract** We study the Riemannian aspect and the Hilbert-Einstein gravitational action of the non-commutative geometry underlying the Connes-Lott construction of the action functional of the standard model. This geometry involves a two-sheeted, Euclidian space-time. We show that if we require the space of forms to be locally isotropic and the Higgs scalar to be dynamical, then the Riemannian metrics on the two sheets of Euclidian space-time must be identical. We also show that the distance function between the two sheets is determined by a single, real scalar field whose VEV sets the weak scale.

# 1 Introduction

Among recent ideas on the structure of space-time at short distance scales the proposal that space-time is an aspect of a non-commutative (metric) space has the appealing feature that it allows one to develop a natural geometric setting for the standard model [1,2]. In particular, the Higgs sector finds a natural geometrical interpretation.

The non-commutative space underlying the standard model is built over a Euclidian space-time consisting of two copies of Euclidian space $\mathbb{E}^4$. The weak scale turns out to be given by the inverse of the distance between the two copies. If one wants to incorporate gravitational interactions (at least at the classical level) into this formulation of the standard model – as one should do, in principle – the two copies of $\mathbb{E}^4$ must be replaced by a pair of two diffeomorphic Riemannian manifolds which, in general, will have non-vanishing curvatures, and the distance between these two manifolds will be described by a position-dependent, real (more precisely, positive) field, instead of a constant. In ref. [3], we have developed a notion of non-commutative *Riemannian* geometry, following the ideas in [2], and we have studied a simple example built upon a two-sheeted Riemannian space $M_4 \times \mathbb{Z}_2$. The metric and the Levi-Civita connection on the analogue of the cotangent bundle then depend on the choice of a Riemannian metric on $M_4$ and of a real scalar field determining the distance between the two sheets. One can then consider the Hilbert-Einstein action in this example and finds that, besides the usual term proportional to the integral of the scalar curvature over $M_4$, it contains a kinetic term for the scalar field minimally coupled to the metric on $M_4$.

An alternative approach to studying the gravitational sector of the standard model has been proposed [4-7] in which the analogue of the Hilbert-Einstein action is defined as the "Wodzicki residue" of the inverse of the square of a covariant Dirac operator on $M_4 \times \mathbb{Z}_2$, or, equivalently, as the $a_2$-coefficient in the expansion of the trace of the heat kernel associated with the square of the Dirac operator. This definition yields a Hilbert-Einstein action which is the sum of the usual term proportional to the scalar curvature and a term proportional to the square of the scalar field. But there is no kinetic energy term for the scalar field, and thus this field does not propagate. The problem with this approach is that when one couples matter to gravity [8] and then eliminates the non-dynamical scalar field, using its equation of motion, one ends up with a complicated non-linear sigma model involving the matter fields that looks rather meaningless. This

problem can be avoided by defining the gravitational action in terms of the $a_4$-coefficient (instead of the $a_2$-coefficient) in the expansion of the trace of the heat kernel of the square of the Dirac operator. One then arrives at a Weyl-invariant action functional for gravity, and the scalar field has a kinetic term that enables it to propagate [7]. But now the problem arises to see whether there is a mechanism to avoid the spin-2 ghost mode in the metric; a problem that is unsolved.

All these difficulties compell us to return to the strategy proposed in [3] and construct the gravitational action for the standard model in a more conceptual way, using the tools of non-commutative Riemannian geometry. In Sect. 2, we review very briefly Connes' concept of non-commutative geometry and some basic tools of non-commutative Riemannian geometry. Readers who are less mathematically inclined can proceed directly to Section 3. In Sect. 3, we consider the Connes-Lott construction of the action functional of the standard model. We prove that the requirements that there be a non-trivial Higgs field and that the space of differential forms be locally isotropic imply that the Riemannian metrics on the two sheets of Euclidian space-time must be *identical*. We also show that a *single*, real scalar field, setting the weak scale, appears as additional gravitational degrees of freedom, besides those described by the standard space-time metric. We determine the Hilbert-Einstein action and eliminate auxiliary fields by using their equations of motion.

# 2 Some notions and tools of non-commutative Riemannian geometry

The structure of a topological manifold, $M$, is coded into the structure of the abelian algebra, $\mathcal{A}_M$, of complex-valued, continuous functions on $M$. The algebra $\mathcal{A}_M$ is a $^*$-algebra, the $^*$ operation being given by complex conjugation of functions. The manifold $M$ can be viewed as the space of characters of $\mathcal{A}_M$. If $M$ is compact $\mathcal{A}_M$ is unital, i.e., it contains an identity 1, the constant function equal to 1 on $M$. Connes' proposal is to define a compact, non-commutative space in terms of a unital, non-abelian $^*$-algebra $\mathcal{A}$, the "algebra of functions on the non-commutative space"; [2]. A non-commutative space in this sense represents relatively little mathematical structure. In order to develop a differential geometry of non-commutative spaces, one must add more structure; (see [1,2] and [3,9]). Before we describe what structure to add, we briefly review what can be

developed from the data introduced so far: a unital $^*$-algebra $\mathcal{A}$.

Connes [2] defines a $\mathbb{Z}$-graded differential unital algebra of universal forms, $\Omega^\cdot(\mathcal{A})$, over $\mathcal{A}$. The algebra $\Omega^\cdot(\mathcal{A})$ is generated by elements $a \in \mathcal{A}$ of degree 0 and elements $da$, $a \in \mathcal{A}$, of degree 1, with relations $d(a+b) = da + db$, $d(ab) = da\, b + a\, db$ (Leibniz rule), for $a, b$ in $\mathcal{A}$, and $d1 = 0$. An element $\alpha \in \Omega^\cdot(\mathcal{A})$ is said to have degree $n$ if it has the form

$$\alpha \;=\; \sum_j a_j^0\, da_j^1 \ldots da_j^n \quad , \; a_j^i \in \mathcal{A}. \tag{2.1}$$

Let $\Omega^n(\mathcal{A})$ be the vector space of elements of degree $n$. Using the Leibniz rule one verifies that $\Omega^\cdot(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$, with $\Omega^0(\mathcal{A}) = \mathcal{A}$, and $\Omega^i(\mathcal{A})\Omega^j(\mathcal{A}) \subset \Omega^{i+j}(\mathcal{A})$. The differential $d$ on $\Omega^\cdot(\mathcal{A})$ is a linear map of degree 1 defined by

$$d(a_0 da_1 \cdots da_n) \;=\; da_0\, da_1 \cdots da_n \;, \quad a_i \in \mathcal{A}.$$

The identity of $\Omega^\cdot(\mathcal{A})$ is given by $1 \in \mathcal{A} = \Omega^0(\mathcal{A})$. In fact, $\Omega^\cdot(\mathcal{A})$ becomes a $^*$-algebra by defining

$$(da)^* \;=\; -\,da^* \;, \quad (\alpha\beta)^* \;=\; \beta^*\,\alpha^* \;, \quad a \,\in\, \mathcal{A}, \;, \quad \alpha, \beta \,\in\, \Omega^\cdot(\mathcal{A}). \tag{2.2}$$

The cohomology of $\Omega^\cdot(\mathcal{A})$ is trivial.

The $K$-theory of the algebra $\mathcal{A}$ is the study of "vector bundles over the non-commutative space described by $\mathcal{A}$". Inspired by Swan's theorem for vector bundles over compact manifolds, [10], one defines (the space of sections of) a vector bundle, $\mathcal{E}$, over the non-commutative space described by $\mathcal{A}$ as a *finitely generated, projective left $\mathcal{A}$-module*, [2]. A *connection* $\nabla$ on $\mathcal{E}$ is defined to be a linear map

$$\nabla \;:\; \mathcal{E} \;\longrightarrow\; \Omega^1(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{E} \tag{2.3}$$

such that, for any $a \in \mathcal{A}$ and $s \in \mathcal{E}$,

$$\nabla(as) \;=\; da \otimes s \,+\, a\,\nabla s. \tag{2.4}$$

Given $\mathcal{E}$, we define $\Omega^\cdot(\mathcal{E})$ to be the graded left $\Omega^\cdot(\mathcal{A})$-module given by

$$\Omega^\cdot(\mathcal{E}) \;=\; \Omega^\cdot(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{E}, \tag{2.5}$$

where we are using that $\Omega^\cdot(\mathcal{A})$ is a left and right $\mathcal{A}$-module. One calls $\Omega^\cdot(\mathcal{E})$ the space of $\mathcal{E}$-valued, universal forms. A connection $\nabla$ on $\mathcal{E}$ extends uniquely to a linear map of degree one

$$\nabla \;:\; \Omega^\cdot(\mathcal{E}) \;\longrightarrow\; \Omega^\cdot(\mathcal{E})$$

with

$$\nabla(\alpha\sigma) \;=\; d\alpha\sigma \,+\, (-1)^{deg\,\alpha}\,\alpha\,\nabla\,\sigma \tag{2.6}$$

for any homogeneous $\alpha \in \Omega^{\cdot}(\mathcal{A})$ and any $\sigma \in \Omega^{\cdot}(\mathcal{E})$. This observation enables one to define the *curvature* of a connection $\nabla$ by setting

$$R(\nabla) \;:=\; -\,\nabla^2 \;:\; \mathcal{E} \;\longrightarrow\; \Omega^2(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{E}. \tag{2.7}$$

One easily checks (using that $d^2 = 0$) that

$$R(\nabla)\,(as) \;=\; a\;R(\nabla)s \tag{2.8}$$

for any $a \in \mathcal{A}$ and any $s \in \mathcal{E}$. Thus $R(\nabla)$ is an $\mathcal{A}$-*linear* map from $\mathcal{E}$ to $\Omega^2(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{E}$, i.e., a *tensor*. It uniquely extends to an $\mathcal{A}$-linear map from $\Omega^{\cdot}(\mathcal{E})$ to $\Omega^{\cdot}(\mathcal{E})$.

Elements $a \in \mathcal{A}$ are called positive (or non-negative), $a \geq 0$, if they are of the form

$$a \;=\; \sum_i \, b_i^* \, b_i \;, \quad b_i \;\in\; \mathcal{A}.$$

The module $\mathcal{E}$ is called *hermitian* if there is a *hermitian inner product*, $\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E} \to \mathcal{A}$ which is (by definition) a sesquilinear form on $\mathcal{E}$ with the properties that

**i)** $\quad \langle as_1, bs_2\rangle \;=\; a\langle s_1, s_2\rangle b^*, \quad a, b \in \mathcal{A},\; s_1 s_2 \in \mathcal{E}$

**ii)** $\quad \langle s, s\rangle \;\geq\; 0\;, \quad$ for all $\quad s \;\in\; \mathcal{E}$ $\qquad$ (2.9)

**iii)** $\quad$ the map $s \;\mapsto\; \langle s, \cdot\rangle$ from $\mathcal{E}$ to the space, $\mathcal{E}^*$, of $\mathcal{A}-$antilinear functionals on $\mathcal{E}$ is an isomorphism of left $\mathcal{A}-$modules.

A hermitian inner product on $\mathcal{E}$ extends uniquely to a sesquilinear form

$$\langle\cdot,\cdot\rangle \;:\; \Omega^{\cdot}(\mathcal{E}) \times \Omega^{\cdot}(\mathcal{E}) \longrightarrow \Omega^{\cdot}(\mathcal{A})$$

on $\Omega^{\cdot}(\mathcal{E})$ with the property that

$$\langle\alpha\sigma_1, \beta\sigma_2\rangle \;\;=\;\; \alpha\,\langle\sigma_1, \sigma_2\rangle\,\beta^* \tag{2.10}$$

for all $\alpha, \beta$ in $\Omega^{\cdot}(\mathcal{A})$, $\sigma_1, \sigma_2 \in \Omega^{\cdot}(\mathcal{E})$. A connection $\nabla$ on $\mathcal{E}$ is called *unitary* (or hermitian) if, for all $s_1, s_2$ in $\mathcal{E}$,

$$d\,\langle s_1, s_2\rangle \;\;=\;\; \langle\nabla\, s_1, s_2\rangle \;\;-\;\; \langle s_1,\,\nabla\, s_2\rangle. \tag{2.11}$$

For homogeneous $\sigma_1$ and $\sigma_2$ in $\Omega^\cdot(\mathcal{E})$, one then has that

$$d\,\langle\sigma_1,\sigma_2\rangle \;=\; \langle\,\nabla\,\sigma_1,\sigma_2\rangle \;-\; (-1)^{\deg\,\sigma_1\,+\,\deg\,\sigma_2}\,\langle\,\sigma_1,\nabla\,\sigma_2\rangle. \qquad (2.12)$$

At this point, it should be noted that the spaces $\Omega^\cdot(\mathcal{A})$ and $\Omega^\cdot(\mathcal{E})$ are "monstrous", and one cannot develop an interesting non-commutative *differential* geometry without introducing further structure. What we are looking for is a notion of *Lipshitz-* or *differentiable structure* on the non-commutative space described by a unital $^*$-algebra $\mathcal{A}$. (Recall that the definition of classical topological manifolds automatically entails that manifolds are Lipshitz. In order to have a notion of non-commutative manifolds, we thus need to introduce a notion of Lipshitz structure.) Such a notion is obtained by considering a *K-cycle* for $\mathcal{A}$, [1,2]: A $K$-cycle for $\mathcal{A}$ is given by the following data:

**i)** a separable Hilbert space $\mathcal{H}$;

**ii)** a (faithful) $^*$-representation $\pi$ of $\mathcal{A}$ by bounded operators on $\mathcal{H}$;

**iii)** a self-adjoint operator $D$ on $\mathcal{H}$, with the properties that $[D,\pi(a)]$ is a bounded operator on $\mathcal{H}$, for all $a\in\mathcal{A}$, and $e^{-\varepsilon D^2}$ is trace-class, for arbitrary $\varepsilon>0$.

<u>Remark</u>: The trace-class property of $e^{-\varepsilon D^2}$, $\varepsilon>0$, expresses the idea that the non-commutative space described by $\mathcal{A}$ is compact. If this space is a "continuum" then $\mathcal{A}$ will be infinite dimensional. If $\pi$ is faithful $\mathcal{H}$ must then be infinite dimensional, too. It then follows that $D$ is unbounded. Property **iii)** fixes a "Lipshitz structure" on the non-commutative space described by $\mathcal{A}$.

Henceforth we call $D$ the *Dirac operator*, following the nomenclature used in the classical case, where $\mathcal{A}=\mathcal{A}_M$. If $\pi$ is faithful we shall write $a$, instead of $\pi(a)$, for the operators on $\mathcal{H}$ corresponding to $a\in\mathcal{A}$. A $K$-cycle $(\mathcal{H},\pi,D)$ is called *even* if there is a unitary involution $\Gamma$ on $\mathcal{H}$ ($\Gamma=\Gamma^*=\Gamma^{-1}$) with the property that $\Gamma a=a\Gamma$, for all $a\in\mathcal{A}$, and $\Gamma D=-D\Gamma$, (i.e. $D$ is odd). Physicists denote $\Gamma$ by $(-1)^F$, $F=$"fermion number". Otherwise, $(\mathcal{H},\pi,D)$ is called *odd.* An odd $K$-cycle $(\mathcal{H},\pi,D)$ determines an even $K$-cycle, $(\tilde{\mathcal{H}},\tilde{\pi},\tilde{D})$, by setting $\tilde{\mathcal{H}}=\mathcal{H}\oplus\mathcal{H}$, $\tilde{\pi}=\pi\oplus\pi$,

$$\tilde{D} \;=\; \begin{pmatrix}0 & D\\ D & 0\end{pmatrix} \quad,\quad \Gamma \;=\; \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad (2.13)$$

A $K$-cycle $(\mathcal{H},\pi,D)$ for $\mathcal{A}$ permits us to define a $^*$-representation, $\pi$, of $\Omega^\cdot(\mathcal{A})$ on $\mathcal{H}$:

$$\pi\,(a^0da^1\cdots da^n) \;=\; a^0[D,a^1]\cdots[D,a^n]. \qquad (2.14)$$

Using the Leibniz rule, one shows that the graded subcomplex $\ker \pi + d \ker \pi$ of $\Omega^{\cdot}(\mathcal{A})$ is a two-sided ideal in $\Omega^{\cdot}(\mathcal{A})$, [2,11]. Thus the quotient

$$\Omega^{\cdot}_{D}(\mathcal{A}) \;:=\; \Omega^{\cdot}(\mathcal{A}) \big/ (\ker \pi \;+\; \ker \pi) \tag{2.15}$$

is a graded differential algebra. We define

$$\begin{aligned} \Omega^{\cdot}_{\pi}(\mathcal{A}) \;&:=\; \pi(\Omega^{\cdot}(\mathcal{A})) \\ \mathrm{Aux} \;&:=\; \pi(d \ker \pi) \\ \Omega^{n}_{\pi,D}(\mathcal{A}) \;&:=\; \Omega^{n}_{\pi}(\mathcal{A}) \big/ \mathrm{Aux}^{n} \\ \Omega^{\cdot}_{\pi,D}(\mathcal{A}) \;&:=\; \bigoplus_{n=0}^{\infty} \Omega^{n}_{\pi,D}(\mathcal{A}) \end{aligned} \tag{2.16}$$

where $\mathrm{Aux}^n$ is the image of all elements of $d \ker \pi$ of degree $n$. One calls Aux the space of “auxiliary fields”, [12]. Note that elements of $\Omega^{\cdot}_{\pi,D}(\mathcal{A})$ are equivalence classes of bounded operators on $\mathcal{H}$ modulo operators in Aux. Whenever there is no danger of confusion we omit reference to the representation $\pi$. We note that $\Omega^{\cdot}_{\pi}(\mathcal{A})$ and Aux are left and right modules over $\mathcal{A}$. Therefore $\Omega^{\cdot}_{\pi,D}(\mathcal{A})$ is a left and right $\mathcal{A}$-module.

Next, we introduce a notion of *integration* on non-commutative spaces. Given $\mathcal{A}$ and a $K$-cycle $(\mathcal{H}, \pi, D)$ for $\mathcal{A}$, we define the integral of a form $\alpha \in \Omega^{\cdot}(\mathcal{A})$ by setting

$$⨍ \alpha \;:=\; \operatorname*{Lim_{\omega}}_{\varepsilon \to 0} \; \frac{tr\left(\pi(\alpha)\exp(-\varepsilon\, D^2)\right)}{tr\left(\exp(-\varepsilon\, D^2)\right)} \tag{2.17}$$

where $\mathrm{Lim}_{\omega}$ denotes a limit defined in terms of some kind of Cesaro mean, see [2]. It must then be checked that $⨍(\cdot)$ is *cyclic*, i.e.,

$$⨍ \alpha\beta \;=\; ⨍ \beta\alpha. \tag{2.18}$$

Formally, this is obvious, and, in the examples we shall consider, eq. (2.18) will be apparent. For general results we refer the reader to [2]. It is clear that $⨍(\cdot)$ defines a non-negative linear functional on $\Omega^{\cdot}(\mathcal{A})$. Thus it determines a positive semi-definite inner product on $\Omega^{\cdot}(\mathcal{A})$:

$$(\alpha, \beta) \;:=\; ⨍ \alpha\beta^{*}, \qquad \alpha, \beta \,\in\, \Omega^{\cdot}(\mathcal{A}). \tag{2.19}$$

The closure of $\Omega^{\cdot}(\mathcal{A})$, modulo the kernel of $(\cdot,\cdot)$, in the norm determined by $(\cdot,\cdot)$ is a Hilbert space denoted by $\tilde{\mathcal{H}} \equiv L^2(\Omega^{\cdot}(\mathcal{A}))$, the Hilbert space of “square integrable forms”.

Clearly, there is a $^*$-representation, $\tilde{\pi}$, of $\Omega^\cdot(\mathcal{A})$, in particular of $\mathcal{A}$, on $\tilde{\mathcal{H}}$. The Hilbert space $\tilde{\mathcal{H}}$ has a filtration into subspaces,

$$\tilde{\mathcal{H}}_0 \subseteq \tilde{\mathcal{H}}_1 \subseteq \cdots \subseteq \tilde{\mathcal{H}}_n \subseteq \cdots \subseteq \tilde{\mathcal{H}} \tag{2.20}$$

where $\tilde{\mathcal{H}}_n$ is the subspace of $\tilde{\mathcal{H}}$ obtained by taking the closure of $\bigcup_{k=0}^{n} \Omega^k(\mathcal{A})$, modulo the kernel of $(\cdot,\cdot)$, in the norm determined by $(\cdot,\cdot)$. We denote by $\bar{\mathcal{A}}$ the weak closure of $\tilde{\pi}(\mathcal{A})$ on $\tilde{\mathcal{H}}_0$. Let $P_D^{(n)}$ denote the orthogonal projection onto $\tilde{\mathcal{H}}_n$, and $P_{d\ker_n}$ the orthogonal projection onto the image of $d\ker\pi|_{\Omega^n(\mathcal{A})}$ in $\tilde{\mathcal{H}}_{n+1}$. Given an element $\alpha \in \Omega^n(\mathcal{A})$, we may define a canonical representative $\alpha^\perp$ in the image of the equivalence class $[\alpha] \in \Omega_D^n(\mathcal{A})$ in $\tilde{\mathcal{H}}_n$ by

$$\alpha^\perp = (1 - P_{d\ker_{n-1}})\,\alpha \in \tilde{\mathcal{H}}_n. \tag{2.21}$$

For $\alpha$ and $\beta$ in $\Omega_D^n(\mathcal{A})$, we set

$$(\alpha,\beta) := (\alpha^\perp, \beta^\perp). \tag{2.22}$$

We define

$$\begin{aligned} \tilde{\Omega}^{\cdot}(\mathcal{A}) &:= \tilde{\pi}\big(\Omega^\cdot(\mathcal{A})\big) \\ \tilde{\Omega}_D^n(\mathcal{A}) &:= \tilde{\pi}\big(\Omega^n(\mathcal{A})\big) \big/ \tilde{\pi}(d\,J_{n-1}) \end{aligned} \tag{2.23}$$

where $J_n$ is the intersection of the kernel of $(\cdot,\cdot)$ with $\Omega^n(\mathcal{A})$. By construction, $\tilde{\Omega}^{\cdot}(\mathcal{A})$ and $\tilde{\Omega}_D^{\cdot}(\mathcal{A})$ are left and right $\mathcal{A}$-modules.

If $\mathcal{A}$ is supposed to describe something like a "finite-dimensional, compact, non-commutative manifold" we must assume that

$$\tilde{\Omega}_D^1(\mathcal{A}) \text{ is a } \underline{\text{finitely generated}},\ \underline{\text{projective left}}\ \mathcal{A}-\underline{\text{module}}. \tag{2.24}$$

We then call $\tilde{\Omega}_D^1(\mathcal{A})$ (the space of sections of) the *cotangent bundle* of (the non-commutative manifold described by) $\mathcal{A}$. One would then expect, moreover, that $\tilde{\Omega}_D^n(\mathcal{A})$ is empty, for all sufficiently large $n$. In "infinite-dimensional" situations, encountered e.g. in string theory, $\tilde{\Omega}_D^1(\mathcal{A})$ will of course not be finitely generated, anymore, and the theory becomes rather tricky.

In order to develop an analogue of Riemannian geometry in the non-commutative case, we should like to equip $\tilde{\Omega}_D^1(\mathcal{A})$ with a *metric*, corresponding to a hermitian inner product

on $\tilde{\Omega}^1_D(\mathcal{A})$. It has been shown in [3,9] that $\tilde{\Omega}^1(\mathcal{A})$ – in fact, $\tilde{\Omega}^{\cdot}(\mathcal{A})$ –*is* equipped with a *canonical metric*, $\langle\cdot,\cdot\rangle_D$, (a generalized hermitian inner product) uniquely determined by $(\mathcal{H},\pi,D)$: For $\alpha$ and $\beta$ in $\tilde{\Omega}^{\cdot}(\mathcal{A})$, we set

$$\langle\alpha,\beta\rangle_D \;:=\; P_D^{(0)}(\alpha\beta^*) \,\in\, \bar{\mathcal{A}}. \tag{2.25}$$

A priori, $P_D^{(0)}(\alpha\beta^*)$ is just a vector in the subspace $\tilde{\mathcal{H}}_0$. However, it turns out that every vector in $\tilde{\mathcal{H}}_0$ uniquely corresponds to an operator on $\tilde{\mathcal{H}}_0$ affiliated with the von Neumann algebra $\bar{\mathcal{A}}$; see [3,9]. This may sound familiar from conformal field theory. The proof follows from the cyclicity of $⨍(\cdot)$. One then easily verifies that $\langle\cdot,\cdot\rangle_D$ satisfies (2.9) (with the possible exception of (**iii)**) and is non-degenerate on $\tilde{\Omega}^{\cdot}(\mathcal{A})$; see [3]. Thus, it defines a "generalized" hermitian inner product, or *metric* on $\tilde{\Omega}^{\cdot}(\mathcal{A})$. If $J_0=\{0\}$, it follows, as a special case, that the cotangent bundle, $\tilde{\Omega}^1_D(\mathcal{A})$, carries a canonical metric. In the sequel, we shall assume that property **iii)** of (2.9) is satisfied, i.e., for each $\varphi\in\tilde{\Omega}^1_D(\mathcal{A})^*$ there is a $\tilde{\varphi}\in\tilde{\Omega}^1_D(\mathcal{A})$ such that $\varphi(\omega)=\langle\omega,\tilde{\varphi}\rangle_D$ holds for all $\omega\in\tilde{\Omega}^1_D(\mathcal{A})$.

By (2.24), $\tilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle over $\mathcal{A}$. We may thus proceed to study connections on the cotangent bundle $\tilde{\Omega}^1_D(\mathcal{A})$. For each element $[\tilde{\pi}(\alpha)]\in\tilde{\Omega}^n_D(\mathcal{A})$ we define

$$d[\tilde{\pi}(\alpha)] \;:=\; [\tilde{\pi}(d\alpha)] \,\in\, \tilde{\Omega}^{n+1}_D(\mathcal{A}) \tag{2.26}$$

and it follows that $\tilde{\Omega}^{\cdot}_D(\mathcal{A})=\bigoplus_{n=0}^{\infty}\tilde{\Omega}^n_D(\mathcal{A})$ is a differential algebra. Definition (2.26) allows us to carry over the tools developed in eqs. (2.3) through (2.12) to the present context, setting $\mathcal{E}=\tilde{\Omega}^1_D(\mathcal{A})$ and replacing $\Omega^{\cdot}(\mathcal{A})$ by $\tilde{\Omega}^{\cdot}_D(\mathcal{A})$, and $\Omega^{\cdot}(\mathcal{E})$ by

$$\tilde{\Omega}^{\cdot}_D \;:=\; \tilde{\Omega}^{\cdot}_D(\mathcal{A})\otimes_{\mathcal{A}}\tilde{\Omega}^1_D(\mathcal{A}). \tag{2.27}$$

A connection, $\nabla$, on $\tilde{\Omega}^1_D(\mathcal{A})$ is then a $\mathbb{C}$-linear map from $\tilde{\Omega}^1_D(\mathcal{A})$ to $\tilde{\Omega}^1_D(\mathcal{A})\otimes_{\mathcal{A}}\tilde{\Omega}^1_D(\mathcal{A})$ satisfying

$$\nabla(a\tilde{\alpha}) \;=\; da\otimes\tilde{\alpha} \,+\, a\,\nabla\tilde{\alpha} \tag{2.28}$$

for $a\in\mathcal{A}$, $\tilde{\alpha}\in\tilde{\Omega}^1_D(\mathcal{A})$. As above we can extend the definition of $\nabla$ to the space $\tilde{\Omega}^{\cdot}_D$. The Riemann curvature of $\nabla$ is then defined by

$$R(\nabla) \;=\; -\nabla^2. \tag{2.29}$$

We shall say that $\nabla$ is *unitary* if the formal equation

$$d\langle\tilde{\alpha},\tilde{\beta}\rangle_D \;=\; \langle\nabla\,\tilde{\alpha},\tilde{\beta}\rangle_D \,-\, \langle\tilde{\alpha},\nabla\,\tilde{\beta}\rangle_D \tag{2.30}$$

is satisfied, for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Omega}^1_D(\mathcal{A})$, in a sense to be made precise in more specific contexts. (The problem in interpreting eq. (2.30) is that $\langle \tilde{\alpha}, \tilde{\beta} \rangle_D$ need not, in general, be an element of the algebra $\mathcal{A}$ – it belongs to the weak closure of $\mathcal{A}$ on $\tilde{\mathcal{H}}_0$ – so the definition of the differential of $\langle \tilde{\alpha}, \tilde{\beta} \rangle_D$ is not, a priori, clear.)

Since, by (2.24), $\tilde{\Omega}^1_D(\mathcal{A})$ is a finitely generated projective left $\mathcal{A}$-module, there are generators $\{E^A\} \subset \tilde{\Omega}^1_D(\mathcal{A})$, and $\{\varepsilon_A\} \subset \tilde{\Omega}^1_D(\mathcal{A})^*$, $A = 1, \cdots, n$, such that

$$\tilde{\alpha} \;=\; \sum_{i=1}^{n} \varepsilon_A\,(\tilde{\alpha})\, E^A \tag{2.31}$$

for any $\tilde{\alpha} \in \tilde{\Omega}^1_D(\mathcal{A})$; see [13]. The Riemann curvature $R(\nabla)$ is an $\mathcal{A}$-linear map from $\tilde{\Omega}^1_D(\mathcal{A})$ to the left $\mathcal{A}$-module $\tilde{\Omega}^2_D(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \tilde{\Omega}^1_D(\mathcal{A})$ and one can thus write $R(\nabla)$ as follows (see [13]):

$$R(\nabla) \;=\; \sum_{A,B} \varepsilon_A \underset{\mathcal{A}}{\otimes} R^A_B \underset{\mathcal{A}}{\otimes} E^B \tag{2.32}$$

where $R^A_B \in \tilde{\Omega}^2_D(\mathcal{A})$. For an arbitrary element $\tilde{\alpha} \in \tilde{\Omega}^1_D(\mathcal{A})$, $R(\nabla)\tilde{\alpha}$ is given by

$$R(\nabla)\tilde{\alpha} \;=\; \sum_{A,B} \varepsilon_A(\tilde{\alpha})\; R^A_B \underset{\mathcal{A}}{\otimes} E^B, \tag{2.33}$$

i.e., it belongs to $\tilde{\Omega}^2_D(\mathcal{A}) \underset{\mathcal{A}}{\otimes} \tilde{\Omega}^1_D(\mathcal{A})$ and is $\mathcal{A}$-linear in $\tilde{\alpha}$. It follows from properties (2.9), of the metric $\langle \cdot, \cdot \rangle_D$ that the map

$$\tilde{\Omega}^1_D(\mathcal{A}) \to \tilde{\Omega}^1_D(\mathcal{A})^* \,, \quad \tilde{\alpha} \mapsto \langle \cdot, \tilde{\alpha}^* \rangle_D,$$

is an isomorphism of right $\mathcal{A}$-modules. Thus, for each $A = 1, \cdots, n$, we can define an element $\tilde{\varepsilon}_A \in \tilde{\Omega}^1_D(\mathcal{A})$ by

$$\varepsilon_A(\tilde{\alpha}) \;=\; \langle \tilde{\alpha}, \tilde{\varepsilon}^*_A \rangle_D \,, \quad \text{for all } \alpha \in \tilde{\Omega}^1_D(\mathcal{A}). \tag{2.34}$$

The *Ricci tensor* associated with the connection $\nabla$ can then be defined *invariantly* by

$$Ric\,(\nabla) \;=\; \sum_{A,B} (P^{(1)}_D - P^{(0)}_D)\,(\tilde{\varepsilon}_A\, R^{A\perp}_B) \underset{\mathcal{A}}{\otimes} E^B \in \tilde{\mathcal{H}}_1 \underset{\mathcal{A}}{\otimes} \tilde{\Omega}^1_D(\mathcal{A}) \tag{2.35}$$

where $R^{A\perp}_B = (1 - P_{d\,J_1})\, R^A_B$, and $P_{dJ_1}$ is the orthogonal projection onto the closure of $dJ_1$. Notice that $\mathcal{A}$ acts on $\tilde{\mathcal{H}}$ from the right due to the cyclicity of $⨍\, (\cdot)$. The *scalar curvature*, $r(\nabla)$, of the connection $\nabla$ can now be defined by

$$r\,(\nabla) \;=\; \sum_{A,B} P^{(0)}_D\, \big((P^{(1)}_D - P^{(0)}_D)(\tilde{\varepsilon}_A\, R^{A\perp}_B)\; E^B\big). \tag{2.36}$$

These definitions are discussed in [14]; (see also [9,11] for a preliminary account.)

Following [3], we define the *torsion*, $T(\nabla)$, of the connection $\nabla$ by

$$T(\nabla) \;=\; d \,-\, m \circ \nabla \tag{2.37}$$

where $m$ is multiplication of forms. One verifies without difficulty that $T(\nabla)$ is an $\mathcal{A}$-linear map from $\tilde{\Omega}^{\cdot}_D$ to $\tilde{\Omega}^{\cdot}_D$, (i.e., $T(\nabla)$ is a tensor). A connection $\nabla$ is called a *Levi-Civita connection* if $\nabla$ is unitary and $T(\nabla) = 0$. In contrast to the classical case, there are "non-commutative Riemannian spaces" $(\mathcal{A}, \mathcal{H}, \pi, D)$ which do not admit any Levi-Civita connection and ones that admit many.

In our calculations in Sect. 3, we shall make use of the non-commutative analogue of the *Cartan structure equations* which were found in [3]. However, since the cotangent bundle $\tilde{\Omega}^1_D(\mathcal{A})$ is, in general, not a free left $\mathcal{A}$-module, we need a slight generalization of these equations. Here, we only state results (for detailed proofs, see [14]).

The components $\Omega^A_{\;B} = \Omega^A_{\;B}(\nabla) \in \tilde{\Omega}^1_D(\mathcal{A})$ of a connection $\nabla$ on $\tilde{\Omega}^1_D(\mathcal{A})$ are defined by

$$\nabla E^A \;=\; -\,\Omega^A_{\;B} \otimes E^B \tag{2.38}$$

where we use the summation convention and drop the subscript $\mathcal{A}$ on the tensor product symbol. Since the generators $\{E^A\}$ of $\tilde{\Omega}^1_D(\mathcal{A})$ are, in general, not linearly independent over $\mathcal{A}$, the coefficients $\Omega^A_{\;B}$ cannot be chosen arbitrarily, and are not unique in general. However, for any matrix $\tilde{\Omega}^A_{\;B} \in \tilde{\Omega}^1_D(\mathcal{A})$, the coefficients

$$\Omega^A_{\;B} \;=\; \varepsilon_C(E^A)\,\tilde{\Omega}^C_{\;D}\,\varepsilon_B(E^D) \;-\; d\,\varepsilon_B(E^A) \tag{2.39}$$

define a connection on $\tilde{\Omega}^1_D(\mathcal{A})$, and every connection is of this form. The components of $T(\nabla)$ and $R(\nabla)$ are defined by

$$T(\nabla)\,E^A \;=\; T^A \in \tilde{\Omega}^2_D\,(\mathcal{A}) \tag{2.40}$$

and

$$R(\nabla)\,E^A \;=\; R^A_{\;B} \otimes E^B, \tag{2.41}$$

where $R^A_{\;B} \in \tilde{\Omega}^2_D(\mathcal{A})$. Notice that the components, $R^A_{\;B}$, of the curvature are not uniquely defined, in general. Combining (2.37) with (2.40) we find that

$$T^A \;=\; dE^A \;+\; \Omega^A_{\;B}\,E^B \;=\; \varepsilon_B(E^A)\,dE^B + \;\tilde{\tilde{\Omega}}{}^A_{\;B}\,E^B \tag{2.42}$$

where $\tilde{\tilde{\Omega}}{}^A_{\ B} = \varepsilon_C(E^A)\,\tilde{\Omega}^C_D\,\varepsilon_B(E^D)$. From (2.28), (2.29) and (2.41) we obtain that

$$\begin{aligned} R^A_B &= d\,\Omega^A_B + \Omega^A_C\,\Omega^C_B \\ &= d\,\tilde{\tilde{\Omega}}{}^A_{\ B} + \tilde{\tilde{\Omega}}{}^A_{\ C}\ \tilde{\tilde{\Omega}}{}^C_{\ B} + d\varepsilon_C(E^A)\,d\varepsilon_B(E^C). \end{aligned} \tag{2.43}$$

Eqs. (2.42) and (2.43) are the non-commutative Cartan equations.

For a Riemannian manifold, the Levi-Civita connection is invariant under all one-parameter groups of isometries. Since, in the non-commutative setting, there are often a lot of Levi-Civita connections, it is useful to look at connections which are also invariant under isometries. A *one-parameter group of isometries* of the "non-commutative Riemannian space" $(\mathcal{A}, \mathcal{H}, \pi, D)$ is a one-parameter group of unitaries $U(t)$ on $\mathcal{H}$ such that, for all $t \in \mathbb{R}$,

$$\begin{aligned} U(t)\,\mathcal{A}\,U(t)^* &= \mathcal{A} \\ [D,\,U(t)] &= 0\,. \end{aligned} \tag{2.44}$$

A connection $\nabla$ is said to be invariant under $U(t)$ if it satisfies

$$\nabla\big(U(t)\,\tilde{\alpha}\,U(t)^*\big) \;=\; \big(U(t)\,\otimes\,U(t)\big)\,\nabla\tilde{\alpha}\,\big(U(t)^*\,\otimes\,U(t)^*\big), \tag{2.45}$$

for any $\tilde{\alpha} \in \tilde{\Omega}^1_D(\mathcal{A})$.

This completes our review of non-commutative Riemannian geometry.

# 3 The non-commutative Riemannian geometry of the standard model

The construction of the standard model in non-commutative geometry [12,1,2,5] requires an appropriate choice of a non-commutative Riemannian space $(\mathcal{A}, \mathcal{H}, \pi, D)$, as defined in the last section. The algebra $\mathcal{A}$ defining the non-commutative space underlying the standard model is chosen to be

$$\mathcal{A} \;=\; (\mathcal{A}_1\,\oplus\,\mathcal{A}_2)\;\otimes\;C^\infty(M_4), \tag{3.1}$$

where $M_4$ is a smooth, compact, four-dimensional Riemannian spin manifold, $\mathcal{A}_1 = \mathbb{M}_2(\mathbb{C})$ is the algebra of complex $2{\times}2$ matrices, and $\mathcal{A}_2 = \mathbb{C}$. (We shall only consider the leptonic

and Higgs sector of the standard model and omit quarks and gluons. They could be included in our analysis, but merely complicate our formulas.) Elements, $a$, of $\mathcal{A}$ are written as

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} , \tag{3.2}$$

where $a_i$ is a $C^\infty$-function on $M_4$ with values in $\mathcal{A}_i$, $i = 1, 2$.

The Hilbert space is defined to be

$$\mathcal{H} = L^2(S_1, dv_1) \oplus L^2(S_2, dv_2), \tag{3.3}$$

where $S_i = S_0 \otimes V_i$, $S_0$ is the usual bundle of Dirac spinors on $M_4$, and $V_i$ is a representation space for $\mathcal{A}_i$, $i = 1, 2$, with $V_1 = \mathbb{C}^2$ and $V_2 = \mathbb{C}$, and $dv_i$ is the volume form corresponding to a Riemannian metric $g_i$ on $M_4$, with $i = 1, 2$. Thus $\mathcal{A}$ acts on sections of $S_1 \oplus S_2$ by left multiplication, and $\mathcal{H}$ is a left $\mathcal{A}$-module of square-integrable $V_1 \oplus V_2$-valued Dirac spinors on $M_4$. The representation $\pi$ of $\mathcal{A}$ is given by

$$\pi = \pi_1 \oplus \pi_2, \tag{3.4}$$

where $\pi_i$ is the representation of $\mathcal{A}_i \otimes C^\infty(M_4)$ on $L^2(S_i, dv_i)$ given by left-multiplication of sections of $S_i$ by elements of $\mathcal{A}_i \otimes C^\infty(M_4)$.

The Dirac operator $D$ is given by

$$D = \begin{pmatrix} \nabla\!\!\!\!/_1 \otimes 1_2 \otimes 1_3 & \gamma^5 \otimes M_{12} \otimes k \\ \gamma^5 \otimes M_{12}^* \otimes k^* & \nabla\!\!\!\!/_2 \otimes 1_3 \end{pmatrix} , \tag{3.5}$$

where $\nabla\!\!\!\!/_i$ is the covariant Dirac operator on $L^2(S_0, dv_i)$; in a coordinate chart, $U$, of $M_4$

$$\nabla\!\!\!\!/_i = e^\mu_{ia} \gamma^a (\partial_\mu + i\omega_{i\mu}), \tag{3.6}$$

where $\{e^\mu_{i\,a}\}$ is a vierbein, i.e., an orthonormal basis of sections of the tangent bundle $TU$, so that

$$\begin{aligned} e^\mu_{i\,a}\, g_{i\mu\nu}\, e^\nu_{i\,b} &= \delta_{ab}, \\ e^\mu_{i\,a}\, \delta^{ab}\, e^\nu_{i\,b} &= g_i^{\mu\nu}; \end{aligned}$$

$\omega_{i\mu} = \frac{1}{4}\, \omega_{\mu ab}(e_i)[\gamma^a, \gamma^b]$ is the corresponding spin connection, with $\omega_{\mu ab}(e_i)$ a solution of the Cartan structure equation

$$T_{ia} = de_{ia} + \sum_b \omega_{ab}\,(e_i)\, e_b = 0,$$

for a unitary connection on $TU$, $i = 1, 2$; and $\{\gamma^a\}_{a=1}^4$ are the anti-hermitian Euclidian Dirac matrices, with $\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}$, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$; (we note that $(\gamma^5)^* = \gamma^5$). Furthermore, $k$ is a $3\times 3$ family mixing matrix, and

$$M_{12} \;=\; \begin{pmatrix} \alpha\,(x) \\ \beta\,(x) \end{pmatrix}, \quad M_{21} \;:=\; M_{12}^*, \tag{3.7}$$

where $\alpha$ and $\beta$ are smooth, complex-valued function on $M_4$; ($M_{12}$ is called a "doublet").

Next, we study the algebra, $\Omega_D^{\cdot}(\mathcal{A})$, of differential forms for $\mathcal{A}$. A 1-form $\rho = \sum_i a_i\, db_i \in \Omega^1(\mathcal{A})$ is represented on $\mathcal{H}$ as the operator

$$\pi(\rho) \;=\; \sum_i\; a_i\,[D, b_i] \;=\; \begin{pmatrix} \gamma^a\, A_{1a} & \gamma_5\, k\, \phi_{12} \\ \gamma_5\, k^*\, \phi_{21} & \gamma^a\, A_{2a} \end{pmatrix}, \tag{3.8}$$

where

$$\begin{aligned} A_{1a} &= e_{1a}^{\mu} \sum_i a_{1i}\, \partial_\mu\, b_{1i}, \\ A_{2a} &= e_{2a}^{\mu} \sum_i a_{2i}\, \partial_\mu\, b_{2i}, \\ \phi_{12} &= \sum_i a_{1i}\, M_{12}\, b_{2i} - M_{12}, \\ \phi_{21} &= \sum_i a_{2i}\, M_{21}\, b_{1i} - M_{21}, \end{aligned} \tag{3.9}$$

where we have assumed, without loss of generality, that

$$\sum a_{1i}\, b_{1i} \;=\; 1,\; \sum a_{2i}\, b_{2i} \;=\; 1\;. \tag{3.10}$$

We also need to understand the space, $\Omega_D^2(\mathcal{A})$, of 2-forms. Since $\pi$ is faithful, $\Omega_D^2(\mathcal{A})$ is isomorphic to $\pi\big(\Omega^2\,(\mathcal{A})\,\big)\big/\mathrm{Aux}^2$, where

$$\mathrm{Aux}^2 \;=\; \Big\{\sum_i [D, a_i][D, b_i] \;:\; \sum_i\; a_i\,[D, b_i] \;=\; 0\Big\}, \tag{3.11}$$

see (2.16). For $\rho = \sum_i a_i\, db_i \in \ker\pi$, $\pi(d\rho) = \sum_i [D, a_i][D, b_i]$ can be evaluated by using eqs. (3.2), (3.5) and (3.9). After some algebra one finds that $\pi(d\rho)$, written as a $2\times 2$ matrix, has the following entries:

$$
\begin{aligned}
\pi\,(d\rho)_{11} &= \not{\partial}\, A_1 - \sum a_{1i}\, \not{\partial}_1^2\, b_{1i} \\
&+ k\, k^* \big( M_{12} (\phi_{21} + M_{21}) + (\phi_{12} + M_{12})\, M_{21} - 2 M_{12}\, M_{21} \\
&- \sum a_{1i}\, [M_{12}\, M_{21}, b_{1i}] \big), \\
\pi\,(d\rho)_{22} &= \not{\partial}\, A_2 - \sum a_{21}\, \not{\partial}_2^2\, b_{2i} \\
&+ k^* k \big( M_{21}\, (\phi_{12} + M_{12}) + (\phi_{21} + M_{21})\, M_{12} - 2 M_{21}\, M_{12} \\
&- \sum a_{2i}\, [M_{21}\, M_{12}, b_{2i}] \big), \\
\pi\,(d\rho)_{12} &= \gamma^5 k \big[ - A_1\, M_{12} + M_{12}\, A_2 - \not{\partial}_1\, (\phi_{12} + M_{12}) \\
&+ \sum a_{1i}\, M_{12}\, (\not{\partial}_1 - \not{\partial}_2)\, b_{2i} + \sum a_{1i}\, (\not{\partial}_1\, M_{12})\, b_{2i} \big], \\
\pi\,(d\rho)_{21} &= \gamma^5 k^* \big[ - A_1\, M_{21} + M_{21}\, A_1 - \not{\partial}_2\, (\phi_{21} + M_{21}) \\
&+ \sum a_{2i}\, M_{21}\, (\not{\partial}_2 - \not{\partial}_1)\, b_{1i} + \sum a_{2i}\, (\not{\partial}_2\, M_{21})\, b_{1i} \big],
\end{aligned}
\tag{3.12}
$$

where $\not{\partial}_i = e^{\mu}_{i\,a}\, \gamma^a\, \partial_\mu$, $i = 1, 2$.

Assuming that $\rho \in \ker \pi$, i.e., $\pi(\rho) = 0$, eqs. (3.12) reduce to

$$
\begin{aligned}
\pi\,(d\rho)_{11} &= -\sum a_{1i}\, \not{\partial}_1^2\, b_{1i} - k\, k^* \sum a_{1i}\, [M_{12}\, M_{21}, b_{1i}], \\
\pi\,(d\rho)_{22} &= -\sum a_{2i}\, \not{\partial}_2^2\, b_{2i} \\
\pi\,(d\rho)_{12} &= \gamma^5 k \big[ \sum a_{1i}\, (\not{\partial}_1\, M_{12})\, b_{2i} - \not{\partial}_1\, M_{12} + \sum a_{1i}\, M_{12}\, (\not{\partial}_1 - \not{\partial}_2)\, b_{2i} \big], \\
\pi\,(d\rho)_{21} &= \gamma^5 k^* \big[ \sum a_{2i}\, (\not{\partial}_2\, M_{21})\, b_{1i} - \not{\partial}_2\, M_{2i} + \sum a_{2i}\, M_{21}\, (\not{\partial}_2 - \not{\partial}_1)\, b_{1i} \big] .
\end{aligned}
\tag{3.13}
$$

Thus $\pi(d\rho)|_{\rho=0}$ is an operator of the form

$$
\begin{pmatrix} X_1 + k\, k^*\, Y_1 & \gamma^5\, k\, \gamma^a\, X_{1a} \\ \gamma^5 k^* \gamma^a X_{2a} & X_2 \end{pmatrix}
\tag{3.14}
$$

where $Y_1$ is an arbitrary function on $M_4$. (We have simplified our notations by omitting writing the identity element of the Clifford algebra and the tensor product symbols.) The structure of the space of auxiliary 2-forms, $\mathrm{Aux}_2$, depends on the properties of $e^{\mu}_{1a} - e^{\mu}_{2a}$ and $\not{\partial}\, M_{12}$. There are three possibilities, namely:

**a)** $X_{1a}$ and $X_{2a}$ are arbitrary functions. Then, the canonical representative, $\omega^\perp$, of a 2-form $[\omega] \in \Omega^2_D(\mathcal{A})$ has vanishing off-diagonal elements and the Higgs field is not dynamical.

**b)** $X_{1a}$ and $X_{2a}$ are neither arbitrary nor identically zero. In this case, the evaluation of $\Omega^2_D(\mathcal{A})$ at a point $p \in M_4$, $\Omega^2_D(\mathcal{A})(p)$, is not everywhere isotropic and its structure may depend on the evaluation point.

**c)** $X_{1a}$ and $X_{2a}$ are identically zero, i.e. auxiliary 2-forms are diagonal. Then the Higgs field is dynamical, $\Omega^2_D(\mathcal{A})(p)$ is everywehere isotropic and its structure doesn't depend on $p$. This is the case we shall consider in the sequel.

Next we compute the constraints implied by the vanishing of $X_{1a}$ and $X_{2a}$. Using (3.13) we see that the condition $\pi(d\rho)_{12}|_{\rho=0} = \pi(d\rho)_{21}|_{\rho=0} = 0$ is equivalent to

$$\begin{aligned} e^\mu_{1a} \; - \; e^\mu_{2a} \; &= \; 0 \\ \sum_i \; a_{1i}\,(\partial\!\!\!/\, M_{12})\,b_{2i} \; - \; \partial\!\!\!/\, M_{12} \; &= \; 0 \\ \sum_i \; a_{2i}\,(\partial\!\!\!/\, M_{21})\,a_{1i} \; - \; \partial\!\!\!/\, M_{21} \; &= \; 0 \end{aligned} \tag{3.15}$$

whenever $\pi(\rho) = 0$. The second equation implies that

$$\partial\!\!\!/\, M_{12} \; = \; \gamma^\mu \, c_\mu \, M_{12} \tag{3.16}$$

for some functions $c_\mu$, and using equation (3.7) one easily proves that

$$M_{12} \; = \; e^{-\sigma} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \tag{3.17}$$

where $\sigma$ is an arbitrary complex valued function and $\alpha_0, \beta_0$ are constant (we exclude the case $\alpha_0 = \beta_0 = 0$ since this would lead to diagonal 2-forms). We can set without loss of generality

$$\alpha_0 \; = \; 0, \quad \beta_0 \; = \; 1, \quad Im\,\sigma \; = \; 0. \tag{3.18}$$

This can be seen by considering the unitarily equivalent $K$-cycle $(\mathcal{H}, \mathcal{A}, UDU^*)$ where

$$U \; = \; \begin{pmatrix} \triangle\beta_0 & -\triangle\alpha_0 & 0 \\ \triangle\bar{\alpha}_0 & \triangle\bar{\beta}_0 & 0 \\ 0 & 0 & e^{i\,Im\,\sigma} \end{pmatrix} \tag{3.19}$$

and $\triangle = (\mid \alpha_0 \mid^2 + \mid \beta_0 \mid^2)^{-1/2}$. The new Dirac operator reads

$$UDU^* \; = \; \begin{pmatrix} \nabla\!\!\!\!/ & k\,\gamma^5\,e^{-Re\,\sigma}\binom{0}{1} \\ k^*\gamma^5\,e^{-Re\,\sigma}\,(0,1) & \nabla\!\!\!\!/ \end{pmatrix} \; + \; \begin{pmatrix} 0 & 0 \\ 0 & i\,\partial\!\!\!/\, Im\,\sigma_2 \end{pmatrix} \tag{3.20}$$

and we can drop the second term since it doesn't contribute to commutators with elements of $\mathcal{A}$. The result of these computations is that the existence of a dynamical Higgs field and local isotropy imply that the distance between the two sheets is described by a single real scalar field, $\sigma$, and that the metrics on them must be identical.

It is not hard to show that, in our case,

$$\int\!\!\!\!\!\!-\,\alpha \;=\; c\Big(\int_{M_4} tr_1\,(\pi(\alpha)_{11})\,dv_1 \;+\; \int_{M_4} tr_2\,(\pi(\alpha)_{22})\,dv_2\Big) \tag{3.21}$$

where $tr_1$ and $tr_2$ are normalized on the generation space such that

$$tr_i\,(k\,k^*) \;=\; 1, \quad tr_i\,(\mathbb{1}_3) \;=\; 1, \quad i \;=\; 1,2 \tag{3.22}$$

and are standard traces on the Clifford algebra and $\mathbb{M}_2(\mathbb{C})$. The constant $c$ is chosen such that $\int\!\!\!\!-\,1 = 1$. Equation (3.21) follows from results in [2]. It follows that

$$(\alpha,\alpha) \;=\; \int\!\!\!\!\!\!-\,\alpha\alpha^* \;=\; 0 \qquad \text{iff } \pi(\alpha) \;=\; 0, \tag{3.23}$$

for all $\alpha \in \Omega^{\bullet}(\mathcal{A})$. This means that $(\cdot,\cdot)$ has a trivial kernel in $\Omega^{\bullet}_{\pi}(\mathcal{A})$ (see (2.16)), and hence the representation $\tilde{\pi}$, defined before eq. (2.20), is a faithful representation of the algebras $\Omega^{\bullet}_{\pi}(\mathcal{A})$ and $\Omega^{\bullet}_{D}(\mathcal{A})$ (the algebra of differential forms) and, in particular of $\mathcal{A}$.

From the fact that the Clifford algebra generated by the Dirac matrices $\gamma^1,\cdots,\gamma^4$ is finite-dimensional one deduces that there exists some $n<\infty$ such that, in eq. (2.20),

$$\tilde{\mathcal{H}}_0 \;\subset\; \tilde{\mathcal{H}}_1 \;\subset\cdots\subset\; \tilde{\mathcal{H}}_n \;=\; \tilde{\mathcal{H}}, \tag{3.24}$$

and hence $\Omega^k_D(\mathcal{A}) = \{0\}$, for $k>n$. From (3.8) and (3.9) we infer that $\tilde{\Omega}^1_D(\mathcal{A})$ is a finitely generated, projective left $\mathcal{A}$-module.

Next, we proceed to determine the Levi-Civita connections, i.e., the unitary torsionless connections. To this aim, we introduce a system of generators of $\tilde{\Omega}^1_D(\mathcal{A})$, $\{E^A\}$, given by

$$E^a \;=\; \gamma^a \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad a \;=\; 1,2,3,4 \tag{3.25}$$

$$E^r \;=\; \gamma^5 \begin{pmatrix} O_2 & k\,e_r \\ -\,k^*\,e_r^{\top} & 0 \end{pmatrix}, \quad r \;=\; 5,6 \tag{3.26}$$

where $e_5 = \binom{1}{0}$ and $e_6 = \binom{0}{1}$. It is then easy to check that the elements $\{\varepsilon_A\} \subset \tilde{\Omega}^1_D(\mathcal{A})^*$ given by

$$\begin{aligned} \varepsilon_a(\omega) \;&=\; \begin{pmatrix} e_a^{\mu}\,\omega_{1\mu} & 0 \\ 0 & e_a^{\mu}\,\omega_{2\mu} \end{pmatrix} \\ \varepsilon_5(\omega) \;&=\; \begin{pmatrix} \frac{\omega_1}{2} & 0 & 0 \\ \frac{\omega_2}{2} & 0 & 0 \\ 0 & 0 & -\,\tilde{\omega}_1 \end{pmatrix} \\ \varepsilon_6(\omega) \;&=\; \begin{pmatrix} 0 & \frac{\omega_1}{2} & 0 \\ 0 & \frac{\omega_2}{2} & 0 \\ 0 & 0 & -\,\tilde{\omega}_2 \end{pmatrix} \end{aligned} \tag{3.27}$$

for any 1-form $\omega$ written as

$$\omega \;=\; \begin{pmatrix} \gamma^\mu\,\omega_{1\mu} & \begin{matrix}\omega_1\\ \omega_2\end{matrix} \\ \tilde\omega_1 \quad \tilde\omega_2 & \gamma^\mu\,\omega_{2\mu} \end{pmatrix}$$

with $\omega_{1\mu}$ a $2\times 2$ matrix, satisfy eq. (2.31), i.e., they build a "dual basis".

We define the connection coefficients by

$$\nabla E^A \;=\; -\,\Omega^A_B \otimes E^B \tag{3.28}$$

where $A, B = 1, \cdots, 6$. The connection coefficients being 1-forms, we write

$$\Omega^A_B \;=\; \begin{pmatrix} \gamma^\mu\,\omega_{1\mu}{}^A_B & k\,\gamma^5\,e^{-\sigma}\,\binom{\omega_1{}^A_B}{\omega_2{}^A_B} \\ k^*\,\gamma^5\,e^{-\sigma}\,(\tilde\omega_1{}^A_B,\ \tilde\omega_2{}^A_B) & \gamma^\mu\,\omega_{2\mu}{}^A_B \end{pmatrix} \tag{3.29}$$

where $\omega_{1\mu}{}^A_B$ is a $2\times 2$ matrix. (In the sequel we shall omit to specify the representation,i.e., we write $\omega$ for $\tilde\pi(\omega)$ for any form $\omega$.) Since the generators, $\{E^A\}$, of $\tilde\Omega^1_D(\mathcal{A})$ are anti-Hermitian, they correspond to real forms. Thus, we assume the matrix elements of the connection coefficients to be real. Since $\tilde\Omega^1_D(\mathcal{A})$ is not a free module, the coefficients $\Omega^A_B$ are not independent. Using eq. (2.39) one gets a large number of constraints. These are listed in the appendix eqs. (A.2) - (A.4). Then, we require the connection to be unitary, i.e., (see eq. (2.30))

$$d\langle E^A, E^B\rangle_D \;=\; -\,\Omega^A_C\,\langle\, E^C, E^B\rangle_D \;+\; \langle E^A, E^C\rangle_D\,(\Omega^A_C)^*. \tag{3.30}$$

The products between the generators are easily computed and one finds

$$\begin{aligned}
\langle E^a, E^b\rangle_D \;&=\; \delta^{ab}\,, \quad a, b \;=\; 1, \cdots, 4 \\
\langle E^a, E^r\rangle_D \;&=\; 0\,, \qquad a \;=\; 1, \cdots, 4,\ r = 5, 6. \\
\langle E^5, E^5\rangle_D \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \langle E^5, E^6\rangle_D \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\langle E^6, E^5\rangle_D \;&=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \langle E^6, E^6\rangle_D \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{3.31}$$

Using eqs. (3.29) and (3.30) one gets the unitarity conditions listed in the appendix, eqs. (A.9) - (A.11). Next, we compute the torsion. The components of the torsion are defined by

$$T^A \;=\; dE^A \;+\; \Omega^A_B\,E^B. \tag{3.32}$$

In order to compute these components, we have to know the differentials of the generators. These are computed as follows: we write $E^5$ and $E^6$ as

$$E^r \;=\; e^\sigma\,[D, m^r]\,, \quad r \;=\; 5,6 \tag{3.33}$$

where $m^r \in \mathcal{A}$ are given by

$$m^5 \;=\; \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad m^6 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.34}$$

One then easily checks that

$$dE^r \;=\; [D, e^\sigma][D, m^r] \;=\; \partial\!\!\!/\,\sigma E^r, \;\; r = 5,6. \tag{3.35}$$

Notice that eq. (3.35) gives already the canonial representative, $(dE^r)^\perp$, of $dE^r$ since the auxiliary 2-forms are diagonal. For completeness we give the general form of the canonical representative, $\omega^\perp$, of a 2-form $\omega \in \tilde\Omega^2_D(\mathcal{A})$,

$$\omega^\perp \;=\; \begin{pmatrix} \gamma^{\mu\nu}\,\omega_{1\mu\nu} & k\,\gamma^\mu\gamma^5 \binom{\omega_{1\mu}}{\omega_{2\mu}} \\ k^*\gamma^\mu\gamma^5\,(\tilde\omega_{1\mu}, \tilde\omega_{2\mu} & \gamma^{\mu\nu}\omega_{2\mu\nu} + (k^*k - 1)\,\omega \end{pmatrix} \tag{3.36}$$

where $\omega_{1\mu\nu}$ is a $2{\times}2$ matrix and $\gamma^{\mu\nu} \;=\; \frac{1}{2}\,[\gamma^\mu, \gamma^\nu]$. Notice that the only effect of the projection $\omega \rightarrow \omega^\perp$ which differs from the classical case is the replacement $k^*k \rightarrow k^*k{-}1$ in the matrix element $\omega_{22}$. Using eqs. (3.32), (3.35) and (3.36) one computes the components of the torsion. The conditions of vanishing torsion are listed in the appendix, eqs. (A.15) and (A.16), where it is shown that the condition $T^A = 0$ makes the $\sigma$-field non-dynamical. Thus, we shall consider unitary connections for which the following weaker condition holds,

$$Tr_k\, T^A \;=\; 0\,, \quad A \;=\; 1, \cdots, 6 \tag{3.37}$$

where $Tr_k$ denotes the trace over the generation space.

The next step is to compute connections which are invariant under isometries of the underlying non-commutative space. Since the classical manifold $M_4$ is not specified, we don't know if it admits any Killing field. Thus, we look for isometries described by a one-parameter group of unitaries, $U(t)$, with constant coefficients. It is easy to prove that the requirements

$$U(t)\,\mathcal{A}\,U(t)^* \;\subset\; \mathcal{A},\;\; [D, U(t)] \;=\; 0 \tag{3.38}$$

imply that $U(t)$ is of the form

$$U(t) \;=\; \begin{pmatrix} e^{-\,it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{it} \end{pmatrix} . \tag{3.39}$$

The transformation properties of the generators, $\{E^A\}$, of $\tilde{\Omega}^1_D(\mathcal{A})$ are then

$$\begin{aligned} U(t)\, E^a\, U(t)^* \;&=\; E^a \;,\; U(t)\, E^6\, U(t)^* \;=\; E^6 \\ U(t)\, E^5\, U(t)^* \;&=\; \gamma^5 \begin{pmatrix} O_2 & k\, e^{-\,2i\varphi}\, e_5 \\ -\,k^*\, e^{2i\varphi}\, e_5^\top & O_1 \end{pmatrix} . \end{aligned} \tag{3.40}$$

The conditions implied by the invariance of the connection under these isometries (see eq. (2.45)) are listed in eqs. (A.19)-(A.21) of the appendix.

Finally we compute the Hilbert-Einstein action for unitary connections which are invariant under isometries and for which $Tr_k T^A(\nabla) = 0$ holds. The components of the curvature are given by (see eq. (2.43))

$$R^A_{\;B} \;=\; d\Omega^A_{\;B} + \Omega^A_{\;C}\Omega^C_{\;B}. \tag{3.41}$$

We write the components of the curvature as

$$R^A_{\;B} \;=\; \begin{pmatrix} \gamma^{\mu\nu}\, R^{(1)\;\;A}_{\mu\nu\;\;B} & k\,\gamma^\mu\,\gamma^5\, e^{-\sigma}\, P_{\mu\;\;B}^{\;\;A} \\ k^*\,\gamma^\mu\,\gamma^5\, e^{-\sigma}\, Q_{\mu\;\;B}^{\;\;A} & \gamma^{\mu\nu}\, R^{(2)\,A}_{\mu\nu\;B} \,+\, (k^*k - 1)\, L^A_{\;B} \end{pmatrix} . \tag{3.42}$$

Explicit formulas for these quantities are given in the appendix, eqs. (A.22) - (A.24). The 1-forms, $\tilde{\varepsilon}_A$, corresponding to the dual forms, $\varepsilon_A$, of eq. (3.27) through the definition (2.34) are easily shown to be

$$\begin{aligned} \tilde{\varepsilon}_a \;&=\; -\,\gamma^a \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 1 \end{pmatrix} \\ \tilde{\varepsilon}_r \;&=\; \gamma^5 \begin{pmatrix} O_2 & -\,k\; e_r \\ \frac{1}{2}\, k^*\, e_r^\top & O \end{pmatrix} . \end{aligned} \tag{3.43}$$

This allows us to compute the matrix elements of the Ricci tensor (see eq. 2.35) and then also the scalar curvature defined in eq. (2.36). The explicit formula for the Ricci tensor is given in eq. (A.25). Then one computes the Hilbert-Einstein action

$$\begin{aligned} ⨍ r \;=\; \int \sqrt{g}\; d^4x\; \Big\{ &\big(e^\nu_a\, e^\mu_b - e^\mu_a\, e^\nu_b\big)\big(Tr\, R^{(1)\;a}_{\mu\nu\;b} + R^{(2)\;a}_{\mu\nu\;b}\big) \;+ \\ &e^{-\sigma}\, \Big[e^\mu_a \big(\frac{1}{2}\, P^{\;5}_{1\mu\;a} \,+\, \frac{1}{2}\, P^{\;6}_{2\mu\;a} \,-\, P^{\;a}_{1\mu\;5} \,-\, P^{\;a}_{2\mu\;6}\big) \;+ \\ &\quad e^\mu_a\, \big(Q^{\;a}_{1\mu\;5} \,+\, Q^{\;a}_{2\mu\;6} \,-\, Q^{\;5}_{1\mu\;a} \,-\, Q^{\;6}_{2\mu\;a}\big)\Big] \;+ \\ &\lambda\, \big(L^5_{\;5} \,+\, L^6_{\;6}\big)\Big\} \end{aligned} \tag{3.44}$$

where $\lambda = Tr\,((k\,k^*)^2) - 1$. A straightforward but lengthy computation shows that the Hilbert-Einstein action, for unitary connections invariant under isometries and such that $Tr_k\, T^A = 0$ holds, is given by

$$\begin{aligned}
⨍ r \;&=\; \int \sqrt{g}\, d^4x\, \Big[-\frac{3}{2}\, R(e) \,+\, e^{-2\sigma}\, \big(2(\omega^a_{2\;a})^2 - 6\,(\omega^a_{2\;b})^2 - 4\,\omega^a_{2\;a} - 4\,\omega^a_{2\;a}\,\omega^5_{1\;6}\big) \\
&+ 6\, e^{-\sigma}\nabla_a\,(e^{\sigma}\,\nabla_a\sigma) \,+\, \lambda\,\big(2(\partial_a\sigma)^2 \,+\, e^{-2\sigma}\,\big((\omega^5_{1\;6})^2 - 2\big)\big)\Big] \qquad (3.45)
\end{aligned}$$

where $R(e)$ is the usual scalar curvature of $M_4$. The fields $\omega^a_{2\;b}$ and $\omega^5_{1\;6}$ can be eliminated by their equations of motion

$$\begin{aligned}
\omega^a_{2\;a} \;&=\; \frac{\lambda}{2}\,\omega^5_{1\;6} \;=\; \frac{4}{1-\frac{8}{\lambda}} \\
\omega^a_{2\;b} \;&=\; 0\;, \quad a \;\neq\; b. \qquad (3.46)
\end{aligned}$$

Inserting (3.46) into (3.45) we get our final result

$$⨍ r \;=\; \int \sqrt{g}\; d^4x\, \Big[-\frac{3}{2}\; R(e) + 2\,(3+\lambda)(\partial_z\sigma)^2 + c(\lambda)\; e^{-2\sigma}\Big] \qquad (3.47)$$

where $c(\lambda) = \frac{\lambda(2\lambda-8)}{8-\lambda}$.
Using eq. (3.22) and the definition of $\lambda$, one proves that $\lambda \in [0,2]$. This implies that $c(\lambda) \in \left[-\frac{4}{3}, 0\right]$ and it follows that the potential is negative-definite.

In reality, this is not the full story. We have only dealt, so far, with the Dirac operator of the leptonic sector. In the standard model, the quark sector must also be introduced. The Dirac operator of the quark sector acts on the space of spinors

$$Q \;=\; \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix} \qquad (3.48)$$

and takes the form:

$$D \;=\; \begin{pmatrix} \partial\!\!\!/ \,\otimes\, 1_2 \otimes\, 1_3 & k'\gamma_5\, e^{-\sigma}\, \binom{0}{1}\;, & k''\, e^{-\sigma}\,\gamma_5\, \binom{-1}{0} \\ k'^*\,\gamma_5\, e^{-\sigma}(0,1)\;\;, & k''^*\, e^{-\sigma}\,\gamma_5(-1,0)\;\;, & \partial\!\!\!/ \,\otimes\, 1_3 \end{pmatrix}\;. \qquad (3.49)$$

Elements of the algebra $\mathcal{A}$ now have the form

$$a \;\rightarrow\; \begin{pmatrix} (a_1)_{mn} & & \\ & a_2 & \\ & & \bar{a}_2 \end{pmatrix}\;. \qquad (3.50)$$

There is no increase in the number of independent components of $\Omega^A_B$, because of the symmetry present in (3.50). The form of the gravitational action in the quark sector will

be the same as (3.47), but now with different coefficients and with dependence on the generation-mixing matrices $k'$ and $k''$ of the $d$ and $u$ quark masses. The total action is of the form

$$
⨍ r \;=\; ⨍ (c_l \, r_l \;+\; c_q \, r_q) \tag{3.51}
$$

where $c_l$ and $c_q$ are arbitrary constants.

The total gravitational action, after eliminating the auxiliary fields, is given by

$$
⨍ r \;=\; \int d^4x \left[ -\frac{1}{2}\, (3c_l + 4c_q) R + \alpha (\nabla_a \sigma)^2 + \beta \, e^{-2\sigma} \right] \tag{3.52}
$$

where

$$
\begin{aligned}
\alpha &= \alpha(c_l, c_q, k_e, k_u, k_d) \\
\beta &= \beta(c_l, c_q, k_e, b_u, k_d)
\end{aligned}
$$

are coupling constants completely determined in terms of $c_l, c_q$ and the electron and up and down quark generation-mixing matrices. We choose the normalizations such that

$$
\begin{aligned}
3c_l \,+\, 4c_q &= \frac{1}{k^2} \\
\alpha &= \frac{1}{2}
\end{aligned} \tag{3.53}
$$

which can be solved for $c_l$ and $c_q$. Then $\beta$ is only a function of $k_e$, $k_u$ and $k_d$. One would hope that $\beta$ will be positive. However, since the gravitational action is non-renormalizable, and in the absence of any understanding of quantum non-commutative geometry, these coefficients do not have any physical significance. The field $\sigma$, being the field whose $VEV$ determines the electroweak scale, plays the role of a link between the gravitational sector and the low-energy sector and may provide a signal of the non-commutative geometric structure of space time.

In a previous paper [8], two of the authors have studied the low-energy effective potential and have shown that the field $\sigma$ acquires a well-determined $VEV$ at the quantum level, for limited ranges of the top quark and Higgs masses. One of the solutions obtained (corresponding to a top quark mass of $\sim 147$ $Gev$) turns out to correspond to a saddle point and is physically unacceptable. The other solution obtained corresponds to a very heavy Higgs mass and lies in the domain, where perturbation theory breaks down, and the formula for the one-loop effective potential cannot be trusted. One point which now

is different from the starting point of [8] is that $\beta$ was zero while now it is non-zero, in general.

This case was included in the analysis of Buchmüller and Busch [15], who found an upper bound on the top quark mass of $\sim 100$ *Gev*. This bound is now experimentally excluded, signaling that nature lies outside the perturbative domain. On the basis of the results found in this paper and in [8, 15], we dare claim that if space-time has a non-commutative structure responsible for the standard model then if the top quark mass is in the suspected energy range around $\sim 170$ *Gev* the one-loop effective action cannot be trusted, and it is likely that the Higgs mass is heavy. To get exact values, one would have to use the exact effective potential which cannot be evaluated perturbatively.

# Appendix

## Constraints equations

Since the module $\tilde{\Omega}^1_D(\mathcal{A})$ is not free, the connection coefficients $\Omega^A_B$ are not arbitrary. In order to compute the constraints, we take an arbitrary matrix of 1-forms $\tilde{\Omega}^A_B$ and we use eq. (2.39)

$$\Omega^A_B \;=\; \varepsilon^A(E_c)\,\tilde{\Omega}^c{}_D\;\varepsilon_B(E^D)\;-\;d\varepsilon_B(E^A). \tag{A.1}$$

Comparing the matrix elements of these 1-forms we get the constraints,

$$\begin{aligned}
\omega_{1\mu}{}^a{}_{5,12} &= \omega_{1\mu}{}^a{}_{5,22} = \omega_{1\mu}{}^a{}_{6,11} = \omega_{1\mu}{}^a{}_{6,21} = 0\\
\omega_{1\mu}{}^a{}_{5,11} &= \omega_{1\mu}{}^a{}_{6,12}\\
\omega_{1\mu}{}^a{}_{5,21} &= \omega_{1\mu}{}^a{}_{6,22}\\
\tilde{\omega}_2{}^a{}_5 &= \tilde{\omega}_1{}^a{}_6 = 0\\
\tilde{\omega}_1{}^a{}_5 &= \tilde{\omega}_2{}^a{}_6
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\omega_{1\mu}{}^5{}_{a,21} &= \omega_{1\mu}{}^5{}_{a,22} = \omega_{1\mu}{}^6{}_{a,11} = \omega_{1\mu}{}^6{}_{a,12} = 0\\
\omega_{1\mu}{}^5{}_{a,11} &= \omega_{1\mu}{}^6{}_{a,21}\\
\omega_{1\mu}{}^5{}_{a,12} &= \omega_{1\mu}{}^6{}_{a,22}\\
\omega_2{}^5{}_a &= \omega_1{}^6{}_a = 0\\
\omega_1{}^5{}_a &= \omega_2{}^6{}_a
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\omega_{1\mu}{}^5{}_{5,12} &= \omega_{1\mu}{}^5{}_{5,21} = \omega_{1\mu}{}^5{}_{5,22} = 0\\
\omega_{1\mu}{}^5{}_{6,11} &= \omega_{1\mu}{}^5{}_{6,21} = \omega_{1\mu}{}^5{}_{6,22} = 0\\
\omega_{1\mu}{}^6{}_{5,11} &= \omega_{1\mu}{}^6{}_{5,12} = \omega_{1\mu}{}^6{}_{5,22} = 0\\
\omega_{1\mu}{}^6{}_{6,11} &= \omega_{1\mu}{}^6{}_{6,12} = \omega_{1\mu}{}^6{}_{6,21} = 0\\
\omega_{1\mu}{}^5{}_{5,11} &= \omega_{1\mu}{}^5{}_{6,12} = \omega_{1\mu}{}^6{}_{5,21} = \omega_{1\mu}{}^6{}_{6,22}\\
\omega_2{}^5{}_5 &= -1,\;\; \tilde{\omega}_2{}^5{}_5 = 1\\
\omega_2{}^5{}_6 &= \omega_1{}^6{}_5 = \omega_1{}^6{}_6 = 0\\
\tilde{\omega}_1{}^5{}_6 &= \tilde{\omega}_2{}^6{}_5 = \tilde{\omega}_1{}^6{}_6 = 0\\
\omega_1{}^5{}_5 &= \omega_2{}^6{}_5,\;\; \tilde{\omega}_1{}^5{}_5 = \tilde{\omega}_2{}^5{}_6\\
\omega_1{}^5{}_6 - 1 &= \omega_2{}^6{}_6,\;\; \tilde{\omega}_2{}^6{}_6 - 1 = \tilde{\omega}_1{}^6{}_5.
\end{aligned} \tag{A.4}$$

At this point it is worth noting that there is another way of computing the constraints. We consider all vanishing linear combinations

$$\alpha_A\, E^A \;=\; 0\,, \quad \alpha_A \,\in\, \mathcal{A}. \tag{A.5}$$

This equation holds if and only if

$$\begin{aligned} \alpha_a &= 0\,, \quad a \;=\; 1,\cdots,4 \\ \alpha_r &= \begin{pmatrix} a_{r,11} & a_{r,12} & 0 \\ a_{r,21} & a_{r,22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{A.6}$$

together with the conditions

$$a_{5,11} \,+\, a_{6,12} \;=\; 0\,, \quad a_{5,21} \,+\, a_{6,22} \;=\; 0. \tag{A.7}$$

Then, we get constraints on the connection coefficients by imposing

$$\alpha_A\, E^A \;=\; 0 \;\Longrightarrow\; \nabla\left(\alpha_A\, E^A\right) \;=\; 0. \tag{A.8}$$

Unfortunately this simpler way gives less constraints than eq. (A.1), because it gives only a minimal set of constraints and leaves arbitrary coefficients which don't contribute to the connection.

**Unitarity conditions**

Here, we give only the equations which are independent of eqs. (A.2) - (A.4)

$$\begin{aligned} \omega_{1\mu}{}^{a}{}_{b,ij} &= -\omega_{1\mu}{}^{b}{}_{a,ji} \\ \omega_{2\mu}{}^{a}{}_{b} &= -\omega_{2\mu}{}^{b}{}_{a} \\ \omega_{1}{}^{a}{}_{b} &= \tilde{\omega}_{1}{}^{b}{}_{a}\,,\; \omega_{2}{}^{a}{}_{b} \;=\; \tilde{\omega}_{2}{}^{b}{}_{a} \end{aligned} \tag{A.9}$$

$$\begin{aligned} 2\,\omega_{1\mu}{}^{a}{}_{5,11} \,+\, \omega_{1\mu}{}^{5}{}_{a,11} &= 0 \\ 2\,\omega_{1\mu}{}^{a}{}_{5,21} \,+\, \omega_{1\mu}{}^{5}{}_{a,12} &= 0 \\ \omega_{2\mu}{}^{a}{}_{5} \,+\, \omega_{2\mu}{}^{5}{}_{a} \;=\; \omega_{2\mu}{}^{a}{}_{6} \,+\, \omega_{2\mu}{}^{6}{}_{a} &= 0 \\ \omega_{1}{}^{a}{}_{5} \;=\; \tilde{\omega}_{1}{}^{5}{}_{a}\,,\; \omega_{2}{}^{a}{}_{5} &= \tilde{\omega}_{2}{}^{5}{}_{a} \\ \omega_{1}{}^{a}{}_{6} \;=\; \tilde{\omega}_{1}{}^{6}{}_{a}\,,\; \omega_{2}{}^{a}{}_{6} &= \tilde{\omega}_{2}{}^{6}{}_{a} \\ 2\,\tilde{\omega}_{1}{}^{a}{}_{5} &= \omega_{1}{}^{5}{}_{a} \end{aligned} \tag{A.10}$$

$$
\begin{aligned}
&\omega_{1\mu}{}^{5}{}_{5} \;=\; \omega_{2\mu}{}^{5}{}_{5} \;=\; \omega_{2\mu}{}^{6}{}_{6} \;=\; 0 \\
&\omega_{2\mu}{}^{5}{}_{6} \;+\; \omega_{2\mu}{}^{6}{}_{5} \;=\; 0 \\
&2\,\tilde{\omega}_{1}{}^{5}{}_{5} \;=\; \omega_{1}{}^{5}{}_{5}\,, \quad 2 \;+\; 2\,\tilde{\omega}_{1}{}^{6}{}_{5} \;=\; \omega_{1}{}^{5}{}_{6}
\end{aligned}
\tag{A.11}
$$

**Conditions of vanishing torsion**

Taking eqs. (A.2) - (A.4) and (A.9) - (A.11) into account the matrix elements of the components of the torsion read,

$$
\begin{aligned}
T^{a}_{11} &= \gamma^{\mu\nu}\,(\partial_\mu\, e^{a}_{\nu} \;+\; e^{b}_{\nu}\,\omega_{1\mu}{}^{a}{}_{b}) \\
T^{a}_{22} &= \gamma^{\mu\nu}\,(\partial_\mu\, e^{a}_{\nu} \;+\; e^{b}_{\nu}\,\omega_{2\mu}{}^{a}{}_{b}) \;+\; 2\,(k^{*}k-1)\, e^{-\sigma}\,\tilde{\omega}_{1}{}^{a}{}_{5} \\
T^{a}_{12} &= k\,\gamma^{\mu}\,\gamma^{5}\, \begin{pmatrix} 2\,\omega_{1\mu}{}^{a}{}_{5,11} \;-\; e^{-\sigma}\, e^{b}_{\mu}\,\omega_{1}{}^{a}{}_{b} \\ 2\,\omega_{1\mu}{}^{a}{}_{5,21} \;-\; e^{-\sigma}\, e^{b}_{\mu}\,\omega_{2}{}^{a}{}_{b} \end{pmatrix} \\
T^{a}_{21} &= -k^{*}\,\gamma^{\mu}\,\gamma^{4}\,(\omega_{2\mu}{}^{a}{}_{5} \;+\; e^{-\sigma}\, e^{b}_{\mu}\,\omega_{1}{}^{b}{}_{a},\; \omega_{2\mu}{}^{a}{}_{6} \;+\; e^{-\sigma}\, e^{b}_{\mu}\,\omega_{2}{}^{b}{}_{a})
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
T^{5}_{11} &= e_5\,\cdot\,(-2\gamma^{\mu\nu}\, e^{a}_{\nu})\,(\omega_{1\mu}{}^{a}{}_{5,11}, \omega_{1\mu}{}^{a}{}_{5,21}) \\
T^{5}_{22} &= -\gamma^{\mu\nu}\, e^{a}_{\nu}\,\omega_{2\mu}{}^{a}{}_{5} \;+\; (k^{*}k-1)\, e^{-\sigma}\,\omega_{1}{}^{5}{}_{5} \\
T^{5}_{12} &= e_5\,\cdot\, k\,\gamma^{\mu}\,\gamma^{5}\,(\partial_\mu\,\sigma - 2e^{-\sigma}\, e^{a}_{\mu}\,\tilde{\omega}_{1}{}^{a}{}_{5}) \\
T^{5}_{21} &= -\,k^{*}\,\gamma^{\mu}\,\gamma^{5}\,(\partial_\mu\,\sigma + e^{-\sigma}\, e^{a}_{\mu}\,\omega^{a}_{1}{}_{5},\; e^{-\sigma}\, e^{a}_{\mu}\,\omega_{2}{}^{a}{}_{5} \;+\; \omega_{2\mu}{}^{5}{}_{6})
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
T^{6}_{11} &= e_6\,\cdot\,(-2\gamma^{\mu\nu}\, e^{a}_{\nu})\,(\omega_{1\mu}{}^{a}{}_{5,11},\; \omega_{1\mu}{}^{a}{}_{5,21}) \\
T^{6}_{22} &= -\gamma^{\mu\nu}\, e^{a}_{\nu}\,\omega_{2\mu}{}^{a}{}_{6} \;+\; (k^{*}k-1)\, e^{-\sigma}\,(\omega_{1}{}^{5}{}_{6} - 1) \\
T^{6}_{12} &= e_6\,\cdot\, k\,\gamma^{\mu}\,\gamma^{5}\,(\partial_\mu\sigma - 2e^{-\sigma}\, e^{a}_{\mu}\,\tilde{\omega}_{1}{}^{a}{}_{5}) \\
T^{6}_{21} &= -k^{*}\,\gamma^{\mu}\,\gamma^{5}\,(e^{-\sigma}\, e^{a}_{\mu}\,\omega_{1}{}^{a}{}_{6} - \omega_{2\mu}{}^{5}{}_{6},\; \partial_\mu\sigma + e^{-\sigma}\, e^{a}_{\mu}\,\omega_{2}{}^{a}{}_{6})\;.
\end{aligned}
\tag{A.14}
$$

Then, imposing $Tr_k T^A = 0$ and using eq. (3.22) we get the following equations

$$
\begin{aligned}
&\gamma^{\mu\nu}\,(\partial_\mu\, e^{a}_{\nu} + e^{b}_{\nu}\,\omega_{1\mu}{}^{a}{}_{b}) \;=\; \gamma^{\mu\nu}\,(\partial_\mu\, e^{a}_{\nu} + e^{b}_{\nu}\,\omega_{2\mu}{}^{a}{}_{b}) \;=\; 0 \\
&2\,\omega_{1\mu}{}^{a}{}_{5,11} - e^{-\sigma}\, e^{b}_{\mu}\,\omega_{1}{}^{a}{}_{b} \;=\; 2\,\omega_{1\mu}{}^{a}{}_{5,21} - e^{-\sigma}\, e^{b}_{\mu}\,\omega_{2}{}^{a}{}_{b} \;=\; 0 \\
&\omega_{2\mu}{}^{a}{}_{5} + e^{-\sigma}\, e^{b}_{\mu}\,\omega_{1}{}^{b}{}_{a} \;=\; \omega_{2\mu}{}^{a}{}_{6} + e^{-\sigma}\, e^{b}_{\mu}\,\omega_{2}{}^{b}{}_{a} \;=\; 0
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
&\gamma^{\mu\nu}\, e^a_\nu\, \omega_{1\mu}{}^a{}_{5,11} \;=\; \gamma^{\mu\nu}\, e^a_\nu\, \omega_{1\mu}{}^a{}_{5,21} \;=\; \gamma^{\mu\nu}\, e^a_\nu\, \omega_{2\mu}{}^a{}_5 \;=\; 0 \\
&\gamma^{\mu\nu}\, e^a_\nu\, \omega_{2\mu}{}^a{}_6 \;=\; 0 \\
&\partial_\mu \sigma - 2\, e^{-\sigma}\, e^a_\mu\, \tilde{\omega}_1{}^a{}_5 \;=\; 0 \\
&\partial_\mu \sigma + e^{-\sigma}\, e^a_\mu\, \omega_1{}^a{}_5 \;=\; 0 \\
&\partial_\mu \sigma + e^{-\sigma}\, e^a_\mu\, \omega_2{}^a{}_6 \;=\; 0 \\
&e^{-\sigma}\, e^a_\mu\, \omega_2{}^a{}_5 + \omega_{2\mu}{}^5{}_6 \;=\; e^a_\mu\, e^{-\sigma}\, \omega_1{}^a{}_6 - \omega_{2\mu}{}^5{}_6 \;=\; 0.
\end{aligned} \tag{A.16}
$$

If we impose $T^A = 0$ we get the additional equations

$$
\tilde{\omega}_1{}^a{}_5 \;=\; \omega_1{}^5{}_5 \;=\; \omega_1{}^5{}_6 - 1 \;=\; 0, \tag{A.17}
$$

and this implies together with eq. (A.16)

$$
\partial_\mu \sigma \;=\; 0. \tag{A.18}
$$

Thus, if we want the $\sigma$-field to be dynamical we should only require $Tr_k\, T^A = 0$.

**Invariance under isometries**

The new constraints coming from the invariance of the connection under isometries are,

$$
\omega_{1\mu}{}^a{}_{b,12} \;=\; 0 \qquad \omega_1{}^a{}_b \;=\; 0 \tag{A.19}
$$

$$
\omega_{1\mu}{}^a{}_{5,11} \;=\; \omega_{2\mu}{}^a{}_5 \;=\; 0, \;\; \omega_2{}^a{}_5 \;=\; \omega_1{}^a{}_6 \;=\; 0 \tag{A.20}
$$

$$
\omega_{2\mu}{}^5{}_6 \;=\; 0, \quad \omega_1{}^5{}_5 \;=\; \omega_1{}^a{}_6 \;=\; 0. \tag{A.21}
$$

**The components of the curvature**

We give explicit formulas for the curvature components.

$$
\begin{aligned}
R^{(1)}_{\mu\nu}{}^A{}_B \;&=\; \frac{1}{2}\left(\partial_\mu\, \omega_{1\nu}{}^A{}_B + \omega_{1\mu}{}^A{}_C\, \omega_{1\nu}{}^C{}_B - (\mu \leftrightarrow \nu)\right) \\
R^{(2)}_{\mu\nu}{}^A{}_B \;&=\; \frac{1}{2}\left(\partial_\mu\, \omega_{2\nu}{}^A{}_B + \omega_{2\mu}{}^A{}_C\, \omega_{2\nu}{}^C{}_B - (\mu \leftrightarrow \nu)\right)
\end{aligned} \tag{A.22}
$$

$$
\begin{aligned}
P_\mu{}^A{}_B \;&=\; \begin{pmatrix} \partial_\mu\, \omega_1{}^A{}_B + \omega_{1\mu}{}^A{}_{B,12} \\ \partial_\mu\, \omega_2{}^A{}_B + \omega_{1\mu}{}^A{}_{B,22} - \omega_{2\mu}{}^A{}_B \end{pmatrix} \;+\; \omega_{1\mu}{}^A{}_C\, \omega^C_B - \omega^A_C\, \omega_{2\mu}{}^C{}_B \\
Q_\mu{}^A{}_B \;&=\; \begin{pmatrix} \partial_\mu\, \tilde{\omega}_1{}^A{}_B - \omega_{1\mu}{}^A{}_{B,21} \\ \partial_\mu\, \tilde{\omega}_2{}^A{}_B - \omega_{1\mu}{}^A{}_{B,22} + \omega_{2\mu}{}^A{}_B \end{pmatrix}^T \;-\; \tilde{\omega}^A_C\, \omega_{1\mu}{}^C{}_B + \omega_{2\mu}{}^A{}_C\, \tilde{\omega}^C_B
\end{aligned} \tag{A.23}
$$

$$
L^A_B \;=\; e^{-2\sigma}\,(\omega_2{}^A{}_B + \tilde{\omega}_2{}^A{}_B + \tilde{\omega}^A_C\, \omega^C_B). \tag{A.24}
$$

**The components of the Ricci tensor**

The Ricci tensor is given by $\mathrm{Ric}_B = (P_D^{(1)} - P_D^{(0)})(\tilde{\varepsilon}_A\, R^A_B)$, see eq. (2.35). One finds

$$
\begin{aligned}
\mathrm{Ric}_{B,11} &= (e^\mu_a\,\gamma^\nu - e^\nu_a\,\gamma^\mu)\, R^{(1)\,a}_{\mu\nu\;\;B} + e^{-\sigma}\,\gamma^\mu\, e_r\, Q^r_{\mu\;B} \\
\mathrm{Ric}_{B,22} &= (e^\mu_a\,\gamma^\nu - e^\nu_a\,\gamma^\mu)\, R^{(2)\,a}_{\mu\nu\;\;B} - e^{-\sigma}\,\gamma^\mu\, \frac{1}{2}\, e^\top_r\, P^r_{\mu\;B} \\
\mathrm{Ric}_{B,12} &= k\,\gamma^5 \left(e^\mu_a\, e^{-\sigma}\, P^a_{\mu\;B} - Tr\left((k^* k)^2 - 1\right)\, e_r\, {L^r}_B\right) \\
\mathrm{Ric}_{B,21} &= k^*\,\gamma^5\, e^{-\sigma}\, e^\mu_a\, Q^a_{\mu\;B}.
\end{aligned}
\tag{A.25}
$$

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